数理解析研究所講究録 1370 巻 2004 年 126-129

On splitting theorems for CAT(0) spaces

宇都宮大学教育学部

保坂 哲也 (Tetsuya Hosaka)

The purpose of this note is to introduce main results of my recent paper [7] about splitting theorems for CAT(0) spaces.

We say that a metric space X is a *geodesic space* if for each $x, y \in X$, there exists an isometry $\xi : [0, d(x,y)] \to X$ such that $\xi(0) = x$ and $\xi(d(x,y)) = y$ (such ξ is called a *geodesic*). Also a metric space X is said to be *proper* if every closed metric ball is compact.

Let X be a geodesic space and let T be a geodesic triangle in X. A *comparison triangle* for T is a geodesic triangle $\overline{T}$ in the Euclidean plane $\mathbb{R}^2$ with same edge lengths as T. Choose two points x and y in T. Let $\bar{x}$ and $\bar{y}$ denote the corresponding points in $\overline{T}$. Then the inequality

$$d(x,y) \leq d_{\mathbb{R}^2}(\bar{x}, \bar{y})$$

is called the *CAT(0)-inequality*, where $d_{\mathbb{R}^2}$ is the natural metric on $\mathbb{R}^2$. A geodesic space X is called a *CAT(0) space* if the CAT(0)-inequality holds for all geodesic triangles T and for all choices of two points x and y in T.

A proper CAT(0) space X can be compactified by adding its ideal boundary ∂X, and $X \cup \partial X$ is a metrizable compactification of X ([2], [4]).

A *geometric* action on a CAT(0) space is an action by isometries which is proper ([2, p.131]) and cocompact. We note that every CAT(0) space on which some group acts geometrically is a proper space ([2, p.132]). Details of CAT(0) spaces and their boundaries are found in [2] and [4].

In [7], we first proved the following splitting theorem which is an extension of Proposition II.6.3 in [2].

Theorem 1. *Suppose that a group $\Gamma = \Gamma_1 \times \Gamma_2$ acts geometrically on a CAT(0) space X. If Γ_1 acts cocompactly on the convex hull $C(\Gamma_1 x_0)$ of some Γ_1-orbit, then there exists a closed, convex, Γ-invariant, quasi-dense subspace $X' \subset X$ such that X' splits as a product $X_1 \times X_2$ and there exist geometric actions of Γ_1 and Γ_2 on X_1 and X_2, respectively. Here each subspace of the form $X_1 \times \{x_2\}$ is the closed convex hull of some Γ_1-orbit.*

Using this theorem, we also proved the following splitting theorem which is an extension of Theorem II.6.21 in [2].

Theorem 2. *Suppose that a group $\Gamma = \Gamma_1 \times \Gamma_2$ acts geometrically on a CAT(0) space X. If the center of Γ is finite, then there exists a closed, convex, Γ-invariant, quasi-dense subspace $X' \subset X$ such that X' splits as a product $X_1 \times X_2$ and the action of $\Gamma = \Gamma_1 \times \Gamma_2$ on $X' = X_1 \times X_2$ is the product action.*

We also showed the following splitting theorem in more general case.

Theorem 3. *Suppose that a group $\Gamma = \Gamma_1 \times \Gamma_2$ acts geometrically on a CAT(0) space X. Then there exist closed convex subspaces X_1, X_2, X_1', X_2' in X such that*

(1) *$X_1 \times X_2'$ and $X_1' \times X_2$ are quasi-dense subspaces of X,*
(2) *X_1' and X_2' are quasi-dense subspaces of X_1 and X_2 respectively,*
(3) *Γ_1 and Γ_2 act geometrically on X_1 and X_2 respectively, and*
(4) *some subgroups of finite index in Γ_1 and Γ_2 act geometrically on X_1' and X_2' respectively.*

A CAT(0) space X is said to have the *geodesic extension property* if every geodesic can be extended to a geodesic line $\mathbb{R} \to X$. Concerning

CAT(0) spaces with the geodesic extension property, we obtained the following theorem as an application of the above splitting theorems.

Theorem 4. *Suppose that a group $\Gamma = \Gamma_1 \times \Gamma_2$ acts geometrically on a CAT(0) space X with the geodesic extension property. Then X splits as a product $X_1 \times X_2$ and there exist geometric actions of Γ_1 and Γ_2 on X_1 and X_2, respectively. Moreover if Γ has finite center, then Γ preserves the splitting, i.e., the action of $\Gamma = \Gamma_1 \times \Gamma_2$ on $X = X_1 \times X_2$ is the product action.*

Let Y be a compact geodesic space of non-positive curvature. Then the universal covering X of Y is a CAT(0) space by the Cartan-Hadamard theorem (cf. [2, p.193, p.237]), and we can think of Y as the quotient $\Gamma \backslash X$ of X, where Γ is the fundamental group of Y acting freely and properly by isometries on X. As an application of Theorem 2, we showed the following splitting theorem which is an extension of Corollary II.6.22 in [2].

Theorem 5. *Let Y be a compact geodesic space of non-positive curvature. Suppose that the fundamental group of Y splits as a product $\Gamma = \Gamma_1 \times \Gamma_2$ and that Γ has trivial center. Then there exists a deformation retract Y' of Y which splits as a product $Y_1 \times Y_2$ such that the fundamental group of Y_i is Γ_i for each $i = 1, 2$.*

A group Γ is called a *CAT(0) group*, if Γ acts geometrically on some CAT(0) space. Theorem 3 implies the following.

Theorem 6. *Γ_1 and Γ_2 are CAT(0) groups if and only if $\Gamma_1 \times \Gamma_2$ is a CAT(0) group.*

In [3], Croke and Kleiner proved that there exists a CAT(0) group Γ and CAT(0) spaces X and Y such that Γ acts geometrically on X and Y and the boundaries of X and Y are not homeomorphic. A CAT(0) group Γ is said to be *rigid*, if Γ determines the boundary up to homeomorphism of a CAT(0) space on which Γ acts geometrically. Then we denote $\partial\Gamma$ as the boundary of the rigid CAT(0) group Γ.

A conclusion in [1] implies that if Γ is a rigid CAT(0) group, then $\Gamma \times \mathbb{Z}^n$ is also a rigid CAT(0) group for each $n \in \mathbb{N}$. In [9], Ruane proved that if $\Gamma_1 \times \Gamma_2$ is a CAT(0) group and if Γ_1 and Γ_2 are hyperbolic groups (in the sense of Gromov) then $\Gamma_1 \times \Gamma_2$ is rigid. Concerning rigidity of products of rigid CAT(0) groups, we can obtain the following theorem from Theorem 3 which is an extension of these results.

Theorem 7. *If Γ_1 and Γ_2 are rigid CAT(0) groups, then so is $\Gamma_1 \times \Gamma_2$, and the boundary $\partial(\Gamma_1 \times \Gamma_2)$ is homeomorphic to the join $\partial\Gamma_1 * \partial\Gamma_2$ of the boundaries of Γ_1 and Γ_2.*

References

[1] P. Bowers and K. Ruane, *Boundaries of nonpositively curved groups of the form $G \times \mathbb{Z}^n$*, Glasgow Math. J. **38** (1996), 177–189.

[2] M. R. Bridson and A. Haefliger, *Metric spaces of non-positive curvature*, Springer-Verlag, Berlin, 1999.

[3] C. B. Croke and B. Kleiner, *Spaces with nonpositive curvature and their ideal boundaries*, Topology **39** (2000), 549–556.

[4] E. Ghys and P. de la Harpe (ed), *Sur les Groups Hyperboliques d'apres Mikhael Gromov*, Progr. Math. vol. 83, Birkhäuser, Boston MA, 1990.

[5] M. Gromov, *Hyperbolic groups*, Essays in group theory (S. M. Gersten, ed.), M.S.R.I. Publ. 8, 1987, pp. 75-264.

[6] T. Hosaka, *A splitting theorem for CAT(0) spaces with the geodesic extension property*, Tsukuba J. Math. **27** (2003), 289–293.

[7] ———, *On splitting theorems for CAT(0) spaces and compact geodesic spaces of non-positive curvature*, preprint.

[8] H. B. Lawson and S. T. Yau, *Compact manifolds of nonpositive curvature*, J. Differential Geom. **7** (1972), 211–228.

[9] K. Ruane, *Boundaries of CAT(0) groups of the form $\Gamma = G \times H$*, Topology Appl. **92** (1999), 131–152.

Department of Mathematics, Utsunomiya University, Utsunomiya, 321-8505, Japan

E-mail address: `hosaka@cc.utsunomiya-u.ac.jp`